Canonical Reduction of Symplectic Structures for the Maxwell and Yang-Mills Equations. Part 1

A. Samoilenko$^{(*)}$, A. Prykarpatsky$^{(**)}$,V. Samoylenko$^{(***)}$

*) Institute of Mathematics at NAS, Kyiv 252601, Ukraina
**) Dept. of Applied Mathematics at the AGH, Krakow, 30059 Poland, and Dept. of Physics at the EMU of Gazimagusa, N. Cyprus
(Emails: prykanat@cybergal.com, pryk.anat@excite.com)
*) Dept. of Mechanics and Mathematics at the National University, Kyiv 252601, Ukraina
(Email: vsam@imath.kiev.ua)

Abstract. The canonical reduction algorithm is applied to Maxwell and Yang-Mills equations considered as Hamiltonian systems on some fiber bundles with symplectic and connection structures. The minimum interaction principle proved to have geometric origin within the reduction method devised.

0. Preliminaries

We begin by reviewing the backgrounds of the reduction theory subject to Hamiltonian systems with symmetry on principle fiber bundles. The material is partly available in [1,4], so here will be only sketched but in notation suitable for us.

Let G denote a given Lie group with the unity element $e \in G$ and the corresponding Lie algebra $\mathcal{G} \simeq T_e(G)$. Consider a principal fiber bundle $p : (M, \varphi) \to N$ with the structure group G and base manifold N, on which the Lie group G acts by means of a mapping $\varphi : M \times G \to M$. Namely, for each $g \in G$ there is a group diffeomorphism $\varphi_g : M \to M$, generating for any fixed $u \in M$ the following induced mapping: $\hat{u} : G \to M$, where

$$\hat{u}(g) = \varphi_g(u). \tag{0.1}$$

On the principal fiber bundle $p : (M, \varphi) \to N$ there is assigned a connection $\Gamma(\mathcal{A})$ by means of such a morphism $\mathcal{A}:(T(M), \varphi_{g*}) \to (\mathcal{G}, Ad)$, that for each $u \in M$ a mapping $\mathcal{A}(u) : T_u(M) \to \mathcal{G}$ is a left inverse one to the mapping $\hat{u}_*(e) : \mathcal{G} \to T_u(M)$, that is

$$\mathcal{A}(u)\hat{u}_*(e) = 1. \tag{0.2}$$

Denote by $\Phi_g : T^*(M) \to T^*(M)$ the corresponding lift of the mapping $\varphi_g : M \to M$ for all $g \in G$. If $\alpha^{(1)} \in \Lambda^1(M)$ is the canonical G - invariant 1-form on M, a symplectic structure $\omega^{(2)} \in \Lambda^2(T^*(M))$ given by

$$\omega^{(2)} := d\ pr^*\alpha^{(1)} \tag{0.3}$$

generates the corresponding momentum mapping $l : T^*(M) \to \mathcal{G}^*$, where

$$l(\alpha^{(1)})(u) = \hat{u}_*(e)\alpha^{(1)}(u) \tag{0.4}$$

for all $u \in M$. Remark here that the principal fiber bundle structure $p : (M, \varphi) \to N$ means in part the exactness of the following sequences of mappings:

$$0 \to \mathcal{G} \stackrel{\hat{u}_*(e)}{\to} T_u(M) \stackrel{p_*(u)}{\to} T_{p(u)}(N) = 0, \tag{0.5}$$

that is

$$p_*(u)\hat{u}_*(e) = 0 \tag{0.6}$$

for all $u \in M$. Combining (0.6) with (0.2) and (0.4), one obtains such an embedding:

$$[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u) \in range\ p^*(u) \tag{0.7}$$

for each canonical 1-form $\alpha^{(1)} \in \Lambda^1(M)$ at $u \in M$. The expression (0.7) means of course, that

$$\hat{u}^*(e)[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u) = 0 \tag{0.8}$$

for all $u \in M$. Taking now into account that the mapping $p^*(u) : T^*(N) \to T^*(M)$ is for each $u \in M$ injective, it has the unique inverse mapping $(p^*(u))^{-1}$ upon its image $p^*(u)T^*(N) \subset T^*(M)$. Thereby for each $u \in M$ one can define a morphism $p_{\mathcal{A}} : (T^*(M), \Phi) \to T^*(N)$ as

$$p_{\mathcal{A}}(u) : \alpha^{(1)} \to (p^*(u))^{-1}[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u). \tag{0.9}$$

Based on the definition (0.9) one can easily check that the diagram

$$\begin{array}{ccc} T^*(M) & \stackrel{p_{\mathcal{A}}}{\to} & T^*(N) \\ pr\downarrow & & \downarrow pr \\ M & \stackrel{p}{\to} & N \end{array} \tag{0.10}$$

is commutative.

Let now an element $\xi \in \mathcal{G}^*$ be $G-$invariant, that is $Ad_g^*\xi = \xi$ for all $g \in G$. Denote also by $p_{\mathcal{A}}^{\xi}$ the restriction of the mapping (0.9) upon the subset $l^{-1}(\xi) \in T^*(M)$, that is $p_{\mathcal{A}}^{\xi} : l^{-1}(\xi) \to T^*(N)$, where for all $u \in M$

$$p_{\mathcal{A}}^{\xi}(u) : l^{-1}(\xi) \to (p^*(u))^{-1}[1 - \mathcal{A}^*(u)\hat{u}^*(e)]l^{-1}(\xi). \tag{0.11}$$

Now one can characterize the structure of the reduced phase space $l^{-1}(\xi)/G$ by means of the following theorem.

Lemma 0.1 *The mapping $p_{\mathcal{A}}^{\xi}(u) : l^{-1}(\xi) \to T^*(N)$ is a principal fiber G-bundle with the reduced space $l^{-1}(\xi)/G$ being diffeomorphic to $T^*(N)$.*

Denote by $< \cdot, \cdot >_{\mathcal{G}}$ the standard Ad-invariant nondegenerate scalar product on $\mathcal{G}^* \times \mathcal{G}$. Based on Lemma 0.1 one derives the following characteristic theorem.

Theorem 0.2 *Given a principal fiber G-bundle with a connection $\Gamma(\mathcal{A})$ and a G-invariant element $\xi \in \mathcal{G}^*$, then each such a connection $\Gamma(A)$ defines a symplectomorphism $\nu_\xi : l^{-1}(\xi)/G \to T^*(N)$ between the reduced phase space $l^{-1}(\xi)/G$ and cotangent bundle $T^*(N)$, where $l : T^*(M) \to \mathcal{G}^*$ is the naturally associated momentum mapping for the group G-action on M. Moreover, the following equality*

$$(p^{\xi}_{\mathcal{A}})(d\ pr^* \beta^{(1)} + pr^*\ \Omega^{(2)}_{\xi}) = d\ pr^* \alpha^{(1)}\Big|_{l^{-1}(\xi)} \tag{0.12}$$

holds for the canonical 1-forms $\beta^{(1)} \in \Lambda^1(N)$ and $\alpha^{(1)} \in \Lambda^1(M)$, where $\Omega^{(2)}_{\xi} :=< \Omega^{(2)}, \xi >_{\mathcal{G}}$ is the ξ-component of the corresponding curvature form $\Omega^{(2)} \in \Lambda^{(2)}(N) \otimes \mathcal{G}$.

Remark 0.3 *As the canonical 2-form $d\ pr^* \alpha^{(1)} \in \Lambda^{(2)}(T^*(M))$ is G-invariant on $T^*(M)$ due to construction, it is evident that its restriction upon the G-invariant submanifold $l^{-1}(\xi) \subset T^*(M)$ will be effectively defined only on the reduced space $l^{-1}(\xi)/G$, that ensures the validity of the equality sign in (0.12).*

As a consequence of Theorem 0.2 one can formulate the following useful enough for applications results.

Theorem 0.4 *Let an element $\xi \in \mathcal{G}^*$ has the isotropy group G_ξ acting on the subset $l^{-1}(\xi) \subset T^*(M)$ freely and properly, so that the reduced phase space $(l^{-1}(\xi)/G\ , \sigma^{(2)}_{\xi})$ is symplectic, where by definition,*

$$\sigma^{(2)}_{\xi} := d\ pr^* \alpha^{(1)}\Big|_{l^{-1}(\xi)}. \tag{0.13}$$

If a principal fiber bundle $p : (M, \varphi) \to N$ has a structure group coinciding with G_ξ , then the reduced symplectic space $(l^{-1}(\xi)/G_\xi, \sigma^{(2)}_{\xi})$ is symplectomorphic to the cotangent symplectic space $(T^*(N), \omega^{(2)}_{\xi})$, where

$$\omega^{(2)}_{\xi} = d\ pr^* \beta^{(1)} + pr^* \Omega^{(2)}_{\xi}, \tag{0.14}$$

and the corresponding symplectomorphism is given by the relation like (0.12).

Theorem 0.5 *In order that two symplectic spaces $(l^{-1}(\xi)/G_\xi, \sigma^{(2)}_{\xi})$ and $(T^*(N), d\ pr^* \beta^{(1)})$* were *symplectomorphic, it is necessary and sufficient that the element $\xi \in \ker h$, where for G-invariant element $\xi \in \mathcal{G}^*$ the mapping $h : \xi \to [\Omega^{(2)}_{\xi}] \in H^2(N;\mathbb{Z})$ with $H^2(N;\mathbb{Z})$ being the cohomology class of 2-forms on the manifold N.*

In case when there is given a Lie group G, the tangent space $T(G)$ is also a Lie group isomorphic to the semidirect product $\tilde{G} := G \circledast_{Ad} \mathcal{G}$ of the Lie group G and its Lie algebra $\mathcal{G}$ under the adjoint action Ad of G on $\mathcal{G}$. The Lie algebra $\tilde{\mathcal{G}}$ of $\tilde{G}$ is correspondingly the semidirect product of $\mathcal{G}$ with itself , regarded as a trivial abelian Lie algebra, under the adjoint action ad and has thus the

bracket defined by $[(a_1, m_1), (a_2, m_2)] := ([a_1, a_2], [a_1, m_2] + [a_2, m_1])$ for all $(a_j, m_j) \in \mathcal{G} \circledast_{ad} \mathcal{G}$, $j = \overline{1,2}$. Take now any element $\xi \in \mathcal{G}^*$ and compute its isotropy group G_ξ under the coadjoint action Ad^* of G on $\mathcal{G}^*$, and denote by $\mathcal{G}_\xi$ its Lie algebra. The cotangent bundle $T^*(G)$ is obviously diffeomorphic to $M := G \times \mathcal{G}^*$ on which the Lie group G_ξ acts freely and properly (due to construction) by left translation on the first factor and Ad^* -action on the second one. The corresponding momentum mapping $l : G \times \mathcal{G}^* \to \mathcal{G}_\xi^*$ is obtained as

$$l(h, \alpha) = Ad^*_{h^{-1}} \alpha|_{\mathcal{G}_\xi^*} \tag{0.15}$$

with no critical point. Let now $\eta \in \mathcal{G}^*$ and $\eta(\xi) := \eta|_{\mathcal{G}_\xi^*}$. Therefore the reduced space $(l^{-1}(\eta(\xi))/G_\xi^{\eta(\xi)}, \sigma_\xi^{(2)})$ has to be symplectic due to the well known Marsden-Weinstein reduction theorem [2,5], where $G_\xi^{\eta(\xi)}$ is the isotropy subgroup of the G_ξ -coadjoint action on $\eta(\xi) \in \mathcal{G}_\xi^*$ and the symplectic form $\sigma_\xi^{(2)} := d\, pr^* \alpha^{(1)}\big|_{l^{-1}(\eta(\xi))}$ is naturally induced from the canonical symplectic structure on $T^*(G)$. Define now for $\eta(\xi) \in \mathcal{G}_\xi^*$ the one-form $\alpha^{(1)}_{\eta(\xi)} \in \Lambda^1(G)$ as

$$\alpha^{(1)}_{\eta(\xi)}(h) := R_h^* \eta(\xi), \tag{0.16}$$

where $R_h : G \to G$ is right translation by an element $h \in G$. It is easy to check that the element (0.16) is right G-invariant and left $G_\xi^{\eta(\xi)}$-invariant, thus inducing a one-form on the quotient $N_\xi := G/G_\xi^{\eta(\xi)}$. Denote by $pr^* \alpha^{(1)}_{\eta(\xi)}$ its pull-back to $T^*(N_\xi)$ and form the symplectic manifold $(T^*(N_\xi), d\, pr^* \beta^{(1)} + d\, pr^* \alpha^{(1)}_{\eta(\xi)})$, where $d\, pr^* \alpha^{(1)}_{\eta(\xi)} \in \Lambda^{(2)}(T^*(N_\xi))$ is the canonical symplectic form on $T^*(N_\xi)$. The construction above now can be summarized as the next theorem.

Theorem 0.6 *Let* $\xi, \eta \in \mathcal{G}^*$ and $\eta(\xi) := \eta|_{\mathcal{G}_\xi^*}$ *be fixed. Then the reduced symplectic manifold* $(l^{-1}(\eta(\xi))/G_\xi^{\eta(\xi)}, \sigma_\xi^{(2)})$ *is a symplectic covering of the coadjoint orbit* $Or(\xi, \eta(\xi); \tilde{G})$ *and symplectically embeds onto a subbundle over* $G/G_\xi^{\eta(\xi)}$ *of* $(T^*(G/G_\xi^{\eta(\xi)}), \omega_\xi^{(2)})$, *with* $\omega_\xi^{(2)} := d\; pr^* \beta^{(1)} + d\; pr^* \alpha^{(1)}_{\eta(\xi)} \in \Lambda^2(T^*(G/G_\xi^{\eta(\xi)})$.

The statement above fits into the conditions of Theorem 0.4 if one to define a connection 1-form $\mathcal{A}(g) : T_g(G) \to \mathcal{G}_\xi$ as follows:

$$< \mathcal{A}(g), \xi >_{\mathcal{G}} := R_g^* \eta(\xi) \tag{0.17}$$

for any $\xi \in \mathcal{G}^*$. The expression (0.17) generates a completely horizontal 2-form $d < \mathcal{A}(g), \xi >_{\mathcal{G}}$ on the Lie group G, which gives rise immediately to the symplectic structure $\omega_\xi^{(2)}$ on the reduced phase space $T^*(G/G_\xi^{\eta(\xi)})$.

1 The Maxwell electromagnetic equations .

Under the Maxwell electromagnetic equations we shall understand the following relationships on a cotangent phase space $T^*(N)$ with $N \subset T(D;\mathbb{R}^3)$ being a manifold of vector fields on some almost everywhere smooth enough domain $D \subset \mathbb{R}^3$:

$$\begin{aligned} \partial E/\partial t &= rot B, \qquad \partial B/\partial t = -rot E \\ div E &= \rho, \qquad\quad div B = 0, \end{aligned} \tag{1.1}$$

where $(E, B) \in T^*(N)$ is a vector of electric and magnetic fields and $\rho \in C(D;\mathbb{R})$ is some fixed density function for a smeared out ambient charge.

Aiming to represent equations (1.1) as those on reduced symplectic space, define as in [9] the appropriate configuration space $M \subset \mathcal{T}(D;\mathbb{R}^3)$, with a vector potential field coordinate $A \in M$. The cotangent space $T^*(M)$ may be identified with pairs $(A, Y) \in T^*(M)$, where $Y \in \mathcal{T}^*(D;\mathbb{R}^3)$ is a vector field density in D. On the space $T^*(M)$ there exists the canonical symplectic form $\omega^{(2)} \in \Lambda^2(T^*(M)$, where $\omega^{(2)} := dpr^*\alpha^{(1)}$, and

$$\alpha^{(1)}(A, Y) = \int_D d^3x < Y, dA >:= (Y, dA), \tag{1.2}$$

where by $< \cdot, \cdot >$ we denoted the standard scalar product in $\mathbb{R}^3$ endowed with the measure d^3x, and by $pr : T^*(M) \to M$ we denoted the usual basepoint projection upon the base space M. Define now a Hamiltonian function $H \in \mathcal{D}(T^*(M))$ as

$$H(A, Y) = 1/2((Y, Y) + (rotA, rotA)), \tag{1.3}$$

which is evidently invariant with respect to the following symmetry group G acting on the base manifold M and lifted to $T^*(M)$: for any $\psi \in \mathcal{G} \subset C^{(1)}(D;\mathbb{R})$ and $(A, Y) \in T^*(M)$

$$\varphi_\psi(A) := A + \nabla\psi, \qquad \Phi_\psi(Y) = Y. \tag{1.4}$$

Under the transformation (1.4) the 1-form (1.2) is evidently invariant too since

$$\varphi_\psi^*\alpha^{(1)}(A, Y) = (Y, dA + \nabla d\psi) = -(divY, d\psi) = \alpha^{(1)}(A, Y), \tag{1.5}$$

where we made use of the condition that $d\psi \simeq 0$ in $\Lambda^1(M)$. Thus, the corresponding momentum mapping (0.4) is given as

$$l(A, Y) = -divY \tag{1.6}$$

for all $(A, Y) \in T^*(M)$. If $\rho \in \mathcal{G}^*$, where $\mathcal{G}$ is the corresponding to G Lie algebra, one can define the reduced space $l^{-1}(\rho)/G$, since evidently, the isotropy

group $G_\rho = G$ due to its commutativity. Consider now a principal fiber bundle $p : M \to N$ with the abelian structure group G and a base manifold N taken as

$$N := \{B \in \mathcal{T}(D; \mathbb{R}^3) : divB = 0\}, \tag{1.7}$$

where, by definition

$$p(A) := B = rotA. \tag{1.8}$$

Over this bundle one can build a connection 1-form $\mathcal{A} : T(M) \to \mathcal{G}$, where for all $A \in M$

$$\mathcal{A}(A) \cdot \hat{A}_*(l) = 1, \quad d < \mathcal{A}(A), \rho >_{\mathcal{G}} = \Omega_\rho^{(2)}(B) \tag{1.9}$$

in virtue of commutativity of the Lie algebra $\mathcal{G}$. Then, due to Theorem 0.4 the cotangent manifold $T^*(N)$ is symplectomorphic to the reduced phase space $l^{-1}(\rho)/G \cong \{(B, E) \in T^*(N) : divE = \rho,\ divB = 0\}$ with the canonical symplectic 2-form

$$\omega_\rho^{(2)}(B, E) = (dS, \wedge dB) + d < \mathcal{A}(A), \rho >_{\mathcal{G}}, \tag{1.10}$$

where we put $rot\ S = -E$. The Hamiltonian (1.3) reduces correspondingly to the following classical form:

$$H(B, E) = 1/2((B, B) + (E, E)). \tag{1.11}$$

As a result, the Maxwell equations (1.1) become a Hamiltonian system upon the reduced phase space $T^*(N)$ endowed with the quasicanonical symplectic structure (1.10) and the new Hamiltonian function (1.11).

It is well known that Maxwell equations (1.1) admit a one more canonical symplectic structure on $T^*(N)$, namely

$$\bar{\omega}^{(2)} := (dB, \wedge dE), \tag{1.12}$$

with respect to which they are Hamiltonian too and whose "helicity" conservative Hamiltonian function reads as

$$\bar{H}(B, E) = 1/2((rotE, E) + (rotB, B)), \tag{1.13}$$

where $(B, E) \in T^*(N)$. It easy to see that (1.13) is also an invariant function with respect to the Maxwell equations (1.1). Subject to the Maxwell equations (1.1) a group theoretical interpretation of the symplectic structure (1.12) is still waiting for search.

Notice now that both symplectic structure (1.12) and Hamiltonian (1.13) are invariant with respect to the following abelian group $G^2 = G \times G$ -action:

$$G^2 \ni (\psi, \chi) : (B, E) \to (B + \nabla\psi, E + \nabla\chi) \tag{1.14}$$

for all $(B,E)\in T^*(N)$. Corresponding to (1.14) the momentum mapping $l:T^*(N)\to\mathcal{G}^*\times\mathcal{G}^*$ is calculated out as

$$l(B,E)=(divE,-divB) \tag{1.15}$$

for any $(B,E)\in T^*(N)$. Fixing a value of (1.15) as $l(B,E)=\xi:=(\rho,0)$, that is

$$divE=\rho,\quad divB=0, \tag{1.16}$$

one obtains the reduced phase space $l^{-1}(\xi)/G^2$, since the isotropy subgroup G^2_ξ of the element $\xi\in\mathcal{G}^*\times\mathcal{G}^*$ coincides with entire group G^2. Thus the reduced phase space due to Theorem 0.4 is endowed with the canonical symplectic structure

$$\bar{\omega}^{(2)}(A,Y)=(dY,\wedge dA)+d<\mathcal{A}(A),\xi>_{\mathcal{G}}, \tag{1.17}$$

where $T^*(M)\ni(A,Y)$ are variables constituting the corresponding coordinates upon the cotangent space over an associated fibre bundle $\bar{p}:N\to M$ with a curvature 1-form $\mathcal{A}:T(N)\to\mathcal{G}\times\mathcal{G}$. In virtue of (1.16) one can define the projection map $\bar{p}:N\to M$ as follows:

$$\bar{p}(B):=rot^{-1}B=A \tag{1.18}$$

for any $A\in M\in\mathcal{T}(D;\mathbb{R}^3)$. It is evident that the second condition of (1.16) is satisfied automatically upon the cotangent bundle $T^*(M)$. Subject to the coadjoint variables $Y\in T^*_A(M)$ and $E\in T^*_B(N)$ for all $A\in M$ and $E\in N$ one can easily obtain from the equality $\bar{p}^*\beta^{(1)}=\alpha^{(1)}$ the expression

$$Y=-rotE, \tag{1.19}$$

satisfying the evident condition $divY=0$. The Hamiltonians (1.11) and (1.13) take correspondingly on $T^*(M)$ the forms as

$$\bar{\mathcal{H}}(A,Y)=1/2((rot^3A,A)+(rot^{-1}Y,Y)), \tag{1.20}$$

and

$$\mathcal{H}(A,Y)=1/2((rot^{-1}Y,rot^{-1}Y)+(rotA,rotA)),$$

being obviously invariant too with respect to common evolutions on $T^*(M)$. As was mentioned in [1], the invariant like (1.13) admits the following geometrical interpretation: its quantity is a related with dynamical equations helicity structure, that is a number of closed linkages of the vortex lines present in the ambient phase space.

If one to consider now a motion of a charged particle under a Maxwell field, it is convenient to introduce another fiber bundle structure $p:M\to N$, namely such one that $M=N\times G$, $N:=D\subset\mathbb{R}^3$ and $G:=\mathbb{R}/\{0\}$ being the corre-

sponding (abelian) structure Lie group. An analysis similar to the above gives rise to a reduced upon the space $l^{-1}(\xi)/G \simeq T^*(N)$, $\xi \in \mathcal{G}$, symplectic structure $\omega^{(2)}(q) =< dp, \wedge dq > +d < \mathcal{A}(q,g), \xi >_{\mathcal{G}}$, where $\mathcal{A}(q,g) :=< A(q), dq > +g^{-1}dg$ is a usual connection 1-form on M, with $(q,p) \in T^*(N)$ and $g \in G$. The corresponding canonical Poisson brackets on $T^*(N)$ are easily found to be

$$\{q^i, q^j\} = 0, \quad \{p_j, q^i\} = \delta^i_j, \qquad \{p_i, p_j\} = F_{ji}(q) \tag{1.21}$$

for all $(q,p) \in T^*(N)$. If one introduces a new momentum variable $\tilde{p} := p + A(q)$ on $T^*(N) \ni (q,p)$, it is easy to verify that $\omega^{(2)}_\xi \to \tilde{\omega}^{(2)}_\xi :=< d\tilde{p}, \wedge dq >$, giving rise to the following Poisson brackets [8]:

$$\{q^i, q^j\} = 0, \qquad \{\tilde{p}_j, q^i\} = \delta^i_j, \qquad \{\tilde{p}_i, \tilde{p}_j\} = 0, \tag{1.22}$$

where $i,j = \overline{1,3}$, iff for all $i,j,k = \overline{1,3}$ the standard Maxwell field equations are satisfied on N :

$$\partial F_{ij}/\partial q_k + \partial F_{jk}/\partial q_i + \partial F_{ki}/\partial q_j = 0 \tag{1.23}$$

with the carvature tensor $F_{ij}(q) := \partial A_j/\partial q^i - \partial A_i/\partial q^j$, $i,j = \overline{1,3}$, $q \in N$. Such a construction permits a natural generalization to the case of nonabelian structure Lie group yielding a description of Yang-Mills field equations within the reduction approach.

2. A charged particle phase space structure and Yang-Mills field equations.

As before, we start with defining a phase space M of a particle under a Yang-Mills field in a region $D \subset \mathbb{R}^3$ as $M := D \times G$, where G is a (not in general semisimple) Lie group, acting on M from the right. Over the space M one can define quite naturally a connection $\Gamma(\mathcal{A})$ if to consider the following trivial principal fiber bundle $p : M \to N$, where $N := D$, with the structure group G. Namely, if $g \in G$, $q \in N$, then a connection 1-form on $M \ni (q,g)$ can be written down [1,3,7] as

$$\mathcal{A}(q;g) := g^{-1}(d + \sum_{i=1}^{n} a_i A^{(i)}(q))g, \tag{2.1}$$

where $\{a_i \in \mathcal{G} : i = \overline{1,n}\}$ is a basis of the Lie algebra $\mathcal{G}$ of the Lie group G, and $A_i : D \to \Lambda^1(D)$, $i = \overline{1,n}$, are the Yang-Mills fields in the physical space $D \subset \mathbb{R}^3$.

Now one defines the natural left invariant Liouville form on M as

$$\alpha^{(1)}(q;g) :=< p, dq > + < y, g^{-1}dg >_{\mathcal{G}}, \tag{2.2}$$

where $y \in T^*(G)$ and $< \cdot, \cdot >_{\mathcal{G}}$ denotes as before the usual Ad-invariant nondegenerate bilinear form on $\mathcal{G}^* \times \mathcal{G}$, as evidently $g^{-1}dg \in \Lambda^1(G) \otimes \mathcal{G}$. The main assumption we need to accept for further is that the connection 1-form is

in accordance with the Lie group G action on M. The latter means that the condition

$$R_h^* \mathcal{A}(q;g) = Ad_{h-1}\mathcal{A}(q;g) \tag{2.3}$$

is satisfied for all $(q,g) \in M$ and $h \in G$, where $R_h : G \to G$ means the right translation by an element $h \in G$ on the Lie group G.

Having stated all preliminary conditions needed for the reduction Theorem 0.4 to be applied to our model, suppose that the Lie group G canonical action on M is naturally lifted to that on the cotangent space $T^*(M)$ endowed due to (2.2) with the following G-invariant canonical symplectic structure:

$$\begin{aligned} \omega^{(2)}(q,p;g,y) \quad &: \quad = d\ pr^* \alpha^{(1)}(q,p;g,y) =< dp, \wedge dq > \\ + \quad &< \quad dy, \wedge g^{-1}dg >_{\mathcal{G}} + < ydg^{-1}, \wedge dg >_{\mathcal{G}} \end{aligned} \tag{2.4}$$

for all $(q,p;g,y) \in T^*(M)$. Take now an element $\xi \in \mathcal{G}^*$ and assume that its isotropy subgroup $G_\xi = G$, that is $Ad_h^* \xi = \xi$ for all $h \in G$. In the general case such an element $\xi \in \mathcal{G}^*$ can not exist but trivial $\xi = 0$, as it happens to the Lie group $G = SL_2(\mathbb{R})$. Then one can construct the reduced phase space $l^{-1}(\xi)/G$ symplectomorphic to $(T^*(N), \omega_\xi^{(2)})$, where due to (0.12) for any $(q,p) \in T^*(N)$

$$\begin{aligned} \omega_\xi^{(2)}(q,p) &= < dp, \wedge dq > + < \Omega^{(2)}(q), \xi >_{\mathcal{G}} \\ &= < dp, \wedge dq > + \sum_{s=1}^{n} \sum_{i,j=1}^{3} e_s F_{ij}^{(s)}(q) dq^i \wedge dq^j . \end{aligned} \tag{2.5}$$

In the above we have expanded the element $\mathcal{G}^* \ni \xi = \sum_{i=1}^n e_i a^i$ with respect to the bi-orthogonal basis $\{a^i \in \mathcal{G}^* :< a^i, a_j >_{\mathcal{G}} = \delta_j^i,\ i,j = \overline{1,n}\}$ with $e_i \in \mathbb{R}$, $i = \overline{1,3}$, being some constants, as well we denoted by $F_{ij}^{(s)}(q)$, $i,j = \overline{1,n}$, $s = \overline{1,n}$, the corresponding curvature 2-form $\Omega^{(2)} \in \Lambda^2(N) \otimes \mathcal{G}$ components, that is

$$\Omega^{(2)}(q) := \sum_{s=1}^{n} \sum_{i,j=1}^{3} a_s\ F_{ij}^{(s)}(q) dq^i \wedge dq^j \tag{2.6}$$

for any point $q \in N$. Summarizing calculations accomplished above, we can formulate the following result.

Theorem 2.1 *Suppose a Yang-Mills field (2.1) on the fiber bundle $p : M \to N$ with $M = D \times G$ is invariant with respect to the Lie group G action $G \times M \to M$. Suppose also that an element $\xi \in G^*$ is chosen so that $Ad_G^* \xi = \xi$. Then for the naturally constructed momentum mapping $l : T^*(M) \to G^*$ (being equivariant) the reduced phase space $l^{-1}(\xi)/G \simeq T^*(N)$ is endowed with the canonical symplectic structure (2.5), having the following component-wise Poissoin brackets form:*

$$\{p_i, q^j\}_\xi = \delta_i^j, \quad \{q^i, q^j\}_\xi = 0, \quad \{p_i, p_j\}_\xi = \sum_{s=1}^{n} e_s F_{ji}^{(s)}(q) \tag{2.7}$$

for all $i,j=\overline{1,3}$ *and* $(q,p)\in T^*(N)$.

The correspondingly extended Poisson bracket on the whole cotangent space $T^*(M)$ amounts due to (2.4) into the following set of Poisson relationships:

$$\begin{aligned} \{y_s,y_k\} &= \sum_r^n c_{sk}^r\, y_r, \qquad\qquad \{p_i,q^j\}=\ \delta_i^j\ \ , \\ \{y_s,p_j\} &= 0=\{q^i,q^j\},\ \ \{p_i,p_j\}=\sum_{s=1}^n y_s\, F_{ji}^{(s)}(q), \end{aligned} \tag{2.8}$$

where $i,j=\overline{1,3}$, $c_{sk}^r\in\mathbb{R}$, $s,k,r=\overline{1,n}$, are the structure constants of the Lie algebra $\mathcal{G}$, and we made use of the expansion $A^{(s)}(q)=\sum_{j=1}^3 A_j^{(s)}(q)\, dq^j$ as well made changeable values $e_i:=y_i$, $i=\overline{1,n}$. The result (2.8) can bee seen easily if one to rewrite the expression (2.4) into an extended form as $\omega^{(2)}:=\omega^{(2)}_{ext}$, where $\omega^{(2)}_{ext}:=\omega^{(2)}\big|_{\mathcal{A}_0\to\mathcal{A}}$, $\mathcal{A}_0(g):=g^{-1}dg$, $g\in G$. Thereby one can obtain in virtue of the invariance properties of the connection $\Gamma(\mathcal{A})$ that

$$\omega^{(2)}_{ext}(q,p;u,y)=< dp,\wedge dq>+d<y(g),Ad_{g^{-1}}\mathcal{A}(q;e)>_{\mathcal{G}}$$

$$=< dp,\wedge dq>+<d\ Ad^*_{g^{-1}}y(g),\wedge\mathcal{A}(q;e)>_{\mathcal{G}}=< dp,\wedge dq>+\sum_{s=1}^n dy_s\wedge du^s$$

$$+\sum_{j=1}^3\sum_{s=1}^{nj} A_j^{(s)}(q)dy_s\wedge dq- < Ad^*_{g^{-1}}y(g),\mathcal{A}(q,e)\wedge\mathcal{A}(q,e)>_{\mathcal{G}}$$

$$+\sum_{k\geq s=1}^n\sum_{l=1}^n y_l\ c_{sk}^l\ du^k\wedge du^s+\sum_{k=1}^n\sum_{i\geq j=1} y_sF_{ij}^{(s)}(q)dq^i\wedge dq^j, \tag{2.9}$$

where coordinate points $(q,p;u,y)\in T^*(M)$ are defined as follows: $\mathcal{A}_0(e):=\sum_{s=1}^n du^i\ a_i$, $Ad^*_{g^{-1}}y(g)=y(e):=\sum_{s=1}^n y_s\ a^s$ for any element $g\in G$. Whence one gets right away the Poisson brackets (2.8) plus additional brackets connected with conjugated sets of variables $\{u^s\in\mathbb{R}: s=\overline{1,n}\}\in\mathcal{G}^{\ *}$ and $\{y_s\in\mathbb{R}: s=\overline{1,n}\}\in\mathcal{G}:$

$$\{y_s,u^k\}=\delta_s^k,\ \ \{u^k,q^j\}=0,\ \ \{p_j,u^s\}=A_j^{(s)}(q),\ \ \{u^s,u^k\}=0, \tag{2.10}$$

where $j=\overline{1,3}$, $k,s=\overline{1,n}$, and $q\in N$.

Note here that the suggested above transition from the symplectic structure $\omega^{(2)}$ on $T^*(N)$ to its extension $\omega^{(2)}_{ext}$ on $T^*(M)$ just consists formally in adding to the symplectic structure $\omega^{(2)}$ an exact part, which transforms it into equivalent

one. Looking now at the expressions (2.9) , one can infer immediately that an element $\xi := \sum_{s=1}^{n} e_s a^s \in \mathcal{G}^*$ will be invariant with respect to the Ad^*-action of the Lie group G iff

$$\{y_s, y_k\}|_{y_s=e_s} = \sum_{r=1}^{n} c_{sk}^{r}\ e_r\ \equiv 0 \tag{2.11}$$

identically for all $s,k=\overline{1,n},\ \ j=\overline{1,3}$ and $q\in N$. In this and only this case the reduction scheme elaborated above will go through.

Returning attention to the expression (2.10), one can easily write down the following exact expression:

$$\omega_{ext}^{(2)}(q,p;u,y) = \omega^{(2)}(q,p+\sum_{s=1}^{n} y_s\ A^{(s)}(q)\ ;u,y), \tag{2.12}$$

on the phase space $T^*(M)\ \ni\ (q,p;u,y)$, where we abbreviated for brevity $< A^{(s)}(q), dq >$ as $\sum_{j=1}^{3} A_j^{(s)}(q)\ dq^j$. The transformation like (2.12) was discussed within somewhat different context in article [8] containing also a good background for the infinite dimensional generalization of symplectic structure techniques. Having observed from (2.12) that the simple change of variable

$$\tilde{p} := p + \sum_{s=1}^{n} y_s\ A^{(s)}(q) \tag{2.13}$$

of the cotangent space $T^*(N)$ recasts our symplectic structure (2.9) into the old canonical form (2.4), one obtains that the following new set of Poisson brackets on $T^*(M)\ni(q,\tilde{p};u,y)$:

$$\begin{aligned} \{y_s, y_k\} &= \sum_{r=1}^{n} c_{sk}^{r}\ y_r, \quad \{\tilde{p}_i,\tilde{p}_j\}=0, \quad \{\tilde{p}_i, q^j\}=\delta_i^j, \\ \{y_s, q^j\} &= 0\ = \{\tilde{p}_i,\tilde{p}_j\},\ \{u^s,u^k\}=0, \quad \{y_s,\tilde{p}_j\}=0, \\ \{y_s, q^i\} &= 0, \qquad \{y_s,u^k\}=\delta_s^k, \qquad \{u^s,\tilde{p}_j\}=0, \end{aligned} \tag{2.14}$$

where $k,s=\overline{1,n}$ and $i,j=\overline{1,3}$, holds iff the Yang-Mills equations

$$\partial F_{ij}^{(s)}/\partial q^l + \partial F_{jl}^{(s)}/\partial q^i + \partial F_{li}^{(s)}/\partial q^j \tag{2.15}$$

$$+\sum_{k,r=1}^{n} c_{kr}^{s}(F_{ij}^{(k)}A_l^{(r)} + F_{jl}^{(k)}A_i^{(r)} + F_{li}^{(k)}A_j^{(r)}) = 0$$

are fulfilled for all $s=\overline{1,n}$ and $i,j,l=\overline{1,3}$ on the base manifold N. This effect of complete reduction of Yang-Mills variables from the symplectic structure (2.9) is known in literature [1,8] as the principle of minimal interaction and appeared

to be useful enough for studying different interacting systems as in [9,10]. In part 2 of this work we shall continue a study of reduced symplectic structures connected with infinite dimensional coupled dynamical systems like Yang–Mills -Vlasov, Yang-Mills- Bogoliubov and Yang-Mills-Josephson ones.

3.Acknowledgements.

One of the authors (A.P.) is cordially indebted to Prof. B.A. Kupershmidt for sending a set of inspiring reprints of his articles some of which are cited through this work, and as well to Prof. J. Zagrodzinski for sending his article before publication. Special thanks for nice hospitality and warm research atmosphere are due to the staff of Dept. of Physics at the EMU of N.Cyprus, especially to Profs. M. Halilsoy and A. Istillozlu for valuable discussions of problems under regard.

References

[1] Gillemin V., Sternberg S. On the equations of motion of a classical particle in a Yang-Mills field and the principle of general covariance. Hadronic Journal, 1978, 1, p.1-32.

[2] Abracham R., Marsden J. Foundations of Mechanics, Second Edition, Benjamin Cummings, NY,1978.

[3] Prykarpatsky A., Mykytiuk I. Algebraic integrability of nonlinear dynamical systems on manifolds. Classical and quantum aspects. Kluwer, Dordrecht, 1998.

[4] Kummer J. On the construction of the Reduced phase space of a Hamiltonian system with symmetry. Indiana University Mathem. Journal, 1981, 30,N2, p.281-281.

[5] Ratiu T., Euler-Poisson equations on Lie algebras and the N-dimensional heavy rigid body. Proc. NAS of USA, 1981, 78, N3, p. 1327-1328.

[6] Holm D., Kupershmidt B. Superfluid plasmas: multivelocity nonlinear hydrodynamics of superfluid solutions with charged condensates coupled electromagnetically. Phys. Rev., 1987, 36A, N8, p. 3947-3956.

[7] Moor J.D. Lectures on Seiberg-Witten invariants. Lect. Notes in Math., N1629, Springer, 1996.

[8] Kupershmidt B.A. Infinite-dimensional analogs of the minimal coupling principle and of the Poincare lemma for differential two-forms. Diff. Geom. & Appl. 1992, 2,p. 275-293.

[9] Marsden J., Weinstein A. The Hamiltonian structure of the Maxwell-Vlasov equations. Physica D, 1982, 4, p. 394-406.

[10] Prykarpatsky A., Zagrodzinski J. Dynamical aspects of Josephson type media. Ann. of Inst. H.Poincare, Physique Theorique, v. 70, N5, p. 497-524.